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Wakamatsu tilting modules

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Abstract

We study a generalization of tilting modules to modules of possibly infinite projective dimension, introduced by Wakamatsu in [J. Algebra 114 (1988) 106–114]. In particular, we characterize these modules in terms of suitable subcategories of finitely generated modules and in terms of cotorsion theories.

Tilting theory plays an important role in the representation theory of Artin algebras. In particular, the algebras obtained as endomorphism rings of tilting modules form a central class of Artin algebras, called tilted algebras. For the case of tilting modules of finite projective dimension an interesting correspondence with covariantly (and contravariantly) finite subcategories of the category of finitely generated modules was established in [3] (see also [4]). This has had important applications in the theory of algebraic groups, via work on quasihereditary algebras [17].

A further generalization of tilting modules to modules of possibly infinite projective dimension was made in [19]. The present paper is devoted to studying various aspects of these modules, called Wakamatsu tilting modules as in [10]. In Section 2 we give a generalization of the correspondence between tilting modules and special subcategories established in [3]. In [3] also a correspondence between contravariantly and covariantly finite subcategories was investigated. In particular, associated with a tilting module, there

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are a contravariantly finite subcategory $\mathcal{X}$ and a covariantly finite subcategory $\mathcal{Y}$ such that $(\mathcal{X}, \mathcal{Y})$ is what is called a complete cotorsion theory (see [18] for terminology). In Section 3 we show that, in a similar way, there is a cotorsion theory associated with a Wakamatsu tilting module. Moreover, we associate to a Wakamatsu tilting module also a natural cotorsion pair in the whole category of modules, in this case even a complete one. Conversely, under some additional assumptions we see that a cotorsion pair gives rise to a Wakamatsu tilting module.

It is also interesting to investigate the relationship between Wakamatsu tilting modules and tilting modules. In particular, is any Wakamatsu tilting module of finite projective dimension a tilting module? (see [5]). This is of particular interest because of the close relationship to well known homological conjectures, like the generalized Nakayama conjecture. The last two sections are devoted to a discussion of this topic.

1. Preliminaries

In this section we give basic notations and definitions, and we recall some relevant background material from [3].

Let Λ be an Artin algebra. We denote by $\operatorname{Mod}\Lambda$ the category of left Λ-modules and by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. By D we denote the usual duality in $\operatorname{mod}\Lambda$. Given any module $X \in \operatorname{Mod}\Lambda$, $\operatorname{Add} X$ ($\operatorname{add} X$) is the full subcategory of $\operatorname{Mod}\Lambda$ whose objects are the direct summands of (finite) direct sums of copies of X. Similarly, $\operatorname{Prod} X$ denotes the full subcategory of $\operatorname{Mod}\Lambda$ whose objects are the direct summands of direct products of copies of X.

If $\mathcal{X}$ is a subcategory of $\operatorname{mod}\Lambda$, we denote by $\mathcal{X}^{\perp}$ the subcategory of finitely generated modules N such that $\operatorname{Ext}^i(X, N) = 0$ for any $i > 0$ and for any $X \in \mathcal{X}$. Similarly, ${}^{\perp}X$ denotes the subcategory of finitely generated modules M such that $\operatorname{Ext}^i(M, X) = 0$ for any $i > 0$ and for any $X \in \mathcal{X}$. When $\mathcal{X} = \{T\}$ for a finitely generated module T, these classes are indicated as $T^{\perp}$ and ${}^{\perp}T$.

Let $\mathcal{C}$ be a full subcategory of $\operatorname{mod}\Lambda$ closed under direct sums, direct summands, and isomorphisms. The subcategory $\mathcal{C}$ is called *resolving* if it is closed under extensions and kernels of epimorphisms and if it contains the projective modules. Dually, $\mathcal{C}$ is called *coresolving* if it is closed under extensions and cokernels of monomorphisms and if it contains the injective modules.

Let $M \in \operatorname{mod}\Lambda$, $C_M \in \mathcal{C}$ and $\varphi : C_M \to M$. The morphism φ is a *right $\mathcal{C}$-approximation* of M if the induced morphism $\operatorname{Hom}(C, C_M) \to \operatorname{Hom}(C, M)$ is surjective for any $C \in \mathcal{C}$. A *minimal right $\mathcal{C}$-approximation* of M is a right $\mathcal{C}$-approximation which is also a *right minimal* morphism, i.e., its restriction to any nonzero summand is nonzero. Notice that, if $\mathcal{C}$ is resolving, any right $\mathcal{C}$-approximation is an epimorphism. The subcategory $\mathcal{C}$ is called *contravariantly finite* if any module $M \in \operatorname{mod}\Lambda$ admits a (minimal) right $\mathcal{C}$-approximation. The notions of (*minimal*) *left $\mathcal{C}$-approximation* and of *covariantly finite subcategory* are dually defined; if $\mathcal{C}$ is coresolving, any left $\mathcal{C}$-approximation is a monomorphism. The theory of covariant and contravariant finiteness has been introduced by Auslander and Smalø [4]. An analogous theory has independently been discovered and studied by Enochs [9] and other authors. In their context a right or a left approximation is called a *precover* or a *preenvelope*, respectively. A minimal right or left approximation is called a *cover* or an *envelope*, respectively. Given an arbitrary ring R, a subcategory $\mathcal{C} \subseteq \operatorname{Mod} R$ such that any module admits a $\mathcal{C}$-precover is called a *precover class*. *Preenvelope*, *cover* and *envelope classes* are similarly defined.

Given a subcategory $\mathcal{X} \subseteq \operatorname{mod} \Lambda$, we denote by $\hat{\mathcal{X}}$ the subcategory whose objects are the modules M for which there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with all $X_i \in \mathcal{X}$ for some $n \geqslant 0$. Dually, with $\check{\mathcal{X}}$ we denote the subcategory whose objects are the modules N such that there exists an exact sequence $0 \to N \to X^0 \to X^1 \to \cdots \to X^m \to 0$ with all $X^j \in \mathcal{X}$ for some $m \geqslant 0$.

For any $M \in \operatorname{mod} \Lambda$ we denote by $\operatorname{pd} M$ and $\operatorname{id} M$ the projective and the injective dimension of M, respectively. Given any module $M \in \operatorname{mod} \Lambda$, we may decompose it as $M \cong \bigoplus_{i=1}^{m} M_i^{d_i}$ where each M_i is indecomposable, $d_i > 0$ for any i, and $M_i \ncong M_j$ if $i \neq j$. The module M is called *basic* if $d_i = 1$ for any i. The number of non-isomorphic indecomposable modules occurring in the direct sum decomposition above is uniquely determined and it will be denoted by $\delta(M)$. In particular, $\delta(\Lambda)$ coincides with the rank of the Grothendieck group of Λ.

Following [11], a module $M \in \operatorname{mod} \Lambda$ is called *selforthogonal* if $\operatorname{Ext}^i_\Lambda(M, M) = 0$ for any $i > 0$; the module M is called *exceptional* if it is selforthogonal and $\operatorname{pd} X < \infty$. An exceptional module $T \in \operatorname{mod} \Lambda$ is a *tilting module* if there exists an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \operatorname{add} T$ for any i. Dually, a selforthogonal module $W \in \operatorname{mod} \Lambda$ is a *cotilting* module if $\operatorname{id} W < \infty$ and there exists an exact sequence $0 \to W_n \to \cdots \to W_1 \to W_0 \to \mathrm{D}(\Lambda) \to 0$ with $W_i \in \operatorname{add} W$ for any i. To conclude, let us recall the result giving the correspondence between tilting or cotilting modules and suitable subcategories of $\operatorname{mod} \Lambda$.

1.1. Theorem [3, Theorem 5.5]. *Let $T \in \operatorname{mod} \Lambda$ be a selforthogonal module.*

(a) *$T \mapsto T^\perp$ gives a one-one correspondence between isomorphism classes of basic tilting modules and covariantly finite coresolving subcategories $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ such that $\check{\mathcal{X}} = \operatorname{mod} \Lambda$.*
(b) *$T \mapsto {}^\perp T$ gives a one-one correspondence between isomorphism classes of basic cotilting modules and contravariantly finite resolving subcategories $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ such that $\hat{\mathcal{X}} = \operatorname{mod} \Lambda$.*

2. Wakamatsu tilting modules

Wakamatsu generalized the concept of tilting module in [19]. Following [10], we say that a module $T \in \operatorname{mod} \Lambda$ is a *Wakamatsu tilting module* if T is a selforthogonal module such that there exists a long exact sequence

$$0 \to \Lambda \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots$$

where $T_i \in \operatorname{add} T$ and coker $f_i \in {}^{\perp}T$ for any $i \in \mathbb{N}$. Dually, a *Wakamatsu cotilting module* is a selforthogonal module $T \in \operatorname{mod} \Lambda$ such that there exists a long exact sequence

$$\cdots \xrightarrow{g_2} T^1 \xrightarrow{g_1} T^0 \xrightarrow{g_0} \mathrm{D}\Lambda \to 0$$

where $T^i \in \operatorname{add} T$ and $\ker g_i \in T^{\perp}$ for any $i \in \mathbb{N}$. In this section we give a correspondence between Wakamatsu tilting modules and certain subcategories of mod Λ.

Clearly any tilting module is Wakamatsu tilting and any cotilting module is Wakamatsu cotilting. Moreover the following result shows that Wakamatsu tilting modules actually generalize at the same time both tilting and cotilting modules.

2.1. Proposition ([6, Proposition 2.2], [20, Section 2]). *Let $T \in \operatorname{mod} \Lambda$. Then T is a Wakamatsu tilting module if and only if it is a Wakamatsu cotilting module.*

As we have already mentioned, the aim of the present section is to find a correspondence between Wakamatsu tilting modules and subcategories of mod Λ which generalizes the one stated in Theorem 1.1. Thus, first of all we need to determine a good candidate to generalize the subcategory $T^{\perp}$ for a tilting module T. In particular, we should determine a subcategory $\mathcal{X}$ which coincides with $T^{\perp}$ in the tilting case. In the sequel, given a module M, we denote by $\operatorname{gen}^* M$ the subcategory of all modules N such that there exists a long exact sequence $\cdots \xrightarrow{f_2} M^1 \xrightarrow{f_1} M^0 \xrightarrow{f_0} N \to 0$ where $M^i \in \operatorname{add} M$ and $\operatorname{Ext}^1(M, \ker f_i) = 0$ for any $i \in \mathbb{N}$. We dually define the subcategory $\operatorname{cogen}^* M$.

As is easily proved (see [11]), if $T \in \operatorname{mod} \Lambda$ is a tilting module then $T^{\perp} = T^{\perp} \cap \operatorname{gen}^* T$. This suggests that, given a Wakamatsu tilting module T, the subcategory $T^{\perp} \cap \operatorname{gen}^* T$, also studied by Wakamatsu, is the natural one to consider. So let us investigate the properties of this class. First notice that, if $N \in T^{\perp} \cap \operatorname{gen}^* T$, there exists a long exact sequence

$$\cdots \xrightarrow{f_2} T^1 \xrightarrow{f_1} T^0 \xrightarrow{f_0} N \to 0$$

such that each f_i is a minimal right add T-approximation and $\ker f_i \in T^{\perp} \cap \operatorname{gen}^* T$ for any $i \in \mathbb{N}$. The dual result holds for ${}^{\perp}T \cap \operatorname{cogen}^* T$. In particular from Proposition 2.1 we know that, given a Wakamatsu tilting module $T \in \operatorname{mod} \Lambda$, then $\Lambda \in {}^{\perp}T \cap \operatorname{cogen}^* T$ and $\mathrm{D}(\Lambda) \in T^{\perp} \cap \operatorname{gen}^* T$. Thus there exist two long exact sequences:

$$0 \to \Lambda \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots$$

where $T_i \in \operatorname{add} T$, coker $f_i \in {}^{\perp}T$ and f_i is a minimal left add T-approximation for any i, and

$$\cdots \xrightarrow{g_2} T^1 \xrightarrow{g_1} T^0 \xrightarrow{g_0} \mathrm{D}\Lambda \to 0$$

where $T^j \in \operatorname{add} T$, $\ker g_j \in T^{\perp}$, and g_j is a minimal right add T-approximation for any j. Throughout this section, we denote the modules coker f_i and $\ker g_j$ in the previous long exact sequences by K_i and L_j, respectively. The following result will be useful.

2.2. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module and, for any $i \in \mathbb{N}$, let us define the modules K_i as above. Then $T^{\perp} \cap \operatorname{gen}^* T = (\bigoplus_{i \in \mathbb{N}} K_i \oplus T)^{\perp}$.*

Proof. We have to verify that, given any module $M \in T^{\perp} \cap \operatorname{gen}^* T$, $\operatorname{Ext}^j(K_i, M) = 0$ for $i \geqslant 0$ and $j \geqslant 1$. Let us consider the following exact sequences

$$(*) \quad 0 \to \Lambda \to T_0 \to K_0 \to 0 \qquad (**) \quad 0 \to M_1 \to T_0' \xrightarrow{h} M \to 0,$$

where $T_0, T_0' \in \operatorname{add} T$ and $M_1 \in T^{\perp} \cap \operatorname{gen}^* T$. Any morphism $g : \Lambda \to M$ extends to T_0: indeed, since g lifts to a morphism $g' : \Lambda \to T_0'$ and $K_0 \in {}^{\perp}T$, g' extends to a morphism $g'' : T_0 \to T_0'$. Thus $h \circ g''$ extends g. This proves that $\operatorname{Ext}^1(K_0, M) = 0$ for any $M \in T^{\perp} \cap \operatorname{gen}^* T$. Moreover, by applying the functor $\operatorname{Hom}(K_{t+1}, -)$ to the sequence $(**)$ and the functor $\operatorname{Hom}(-, M_1)$ to the exact sequences $0 \to K_t \to T_t \to K_{t+1} \to 0$, we obtain that $\operatorname{Ext}^1(K_{t+1}, M) \cong \operatorname{Ext}^2(K_{t+1}, M_1) \cong \operatorname{Ext}^1(K_t, M_1)$ for any t. By induction we conclude that $\operatorname{Ext}^1(K_i, M) = 0$ for any $M \in T^{\perp} \cap \operatorname{gen}^* T$, and by dimension shift we get that $\operatorname{Ext}^j(K_i, M) \cong \operatorname{Ext}^{j-i}(\Lambda, M) = 0$.

Conversely, let $M \in (\bigoplus_{i \in \mathbb{N}} K_i \oplus T)^{\perp}$; we want to show that M is generated by T. Let us consider an epimorphism $g : \Lambda^n \to M$ for a suitable $n \in \mathbb{N}$, and the commutative pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^n \arrow[r] \arrow[d, "g"'] & T_0^n \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & K_0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\operatorname{Ext}^1(K_0, M) = 0$, the second row splits and so M is generated by T. Now let $0 \to M_1 \to T_0' \xrightarrow{f_0} M \to 0$ be an exact sequence where f_0 is a right $\operatorname{add} T$-approximation of M. Since $M \in T^{\perp}$, also $M_1 \in T^{\perp}$. Moreover we get the isomorphisms $\operatorname{Ext}^i(K_j, M_1) \cong \operatorname{Ext}^{i+1}(K_{j+1}, M_1) \cong \operatorname{Ext}^i(K_{j+1}, M) = 0$; thus $M_1 \in (\bigoplus_{i \in \mathbb{N}} K_i \oplus T)^{\perp}$ and hence M_1 is generated by T. Repeating the same argument for M_1 and so on, we obtain that $M \in T^{\perp} \cap \operatorname{gen}^* T$. $\square$

In the following propositions we compare some well-known properties of the category $T^{\perp}$ associated to a tilting module T with some analogous properties of the category $W^{\perp} \cap \operatorname{gen}^* W$ associated to a Wakamatsu tilting module W.

2.3. Proposition.

(a) *Let $T \in \operatorname{mod} \Lambda$ be a tilting module. Then*:
  (i) $[{}^{\perp}(T^{\perp})]^{\perp} = T^{\perp}$,

(ii) $T^{\perp} \cap {}^{\perp}(T^{\perp}) = \operatorname{add} T$.

(b) *Let $W \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module and let $\mathcal{X}_W = W^{\perp} \cap \operatorname{gen}^* W$. Then*:

(i) $({}^{\perp}\mathcal{X}_W)^{\perp} = \mathcal{X}_W$,

(ii) $\mathcal{X}_W \cap {}^{\perp}\mathcal{X}_W = \operatorname{add} W$.

Proof. Part (a) is well known.

We obtain the first statement in (b) just observing that, by Proposition 2.2, there exists a module $N \in \operatorname{Mod} \Lambda$ such that $\mathcal{X}_W = N^{\perp}$ and $[{}^{\perp}(N^{\perp})]^{\perp} = N^{\perp}$.

To conclude, let $N \in \mathcal{X}_W \cap {}^{\perp}\mathcal{X}_W$. Since W generates any module in $\mathcal{X}_W$, there exists an exact sequence $0 \to N_1 \to W' \to N \to 0$ where $N_1 \in \mathcal{X}_W$ and $W' \in \operatorname{add} W$. Since $N \in {}^{\perp}\mathcal{X}_W$, this sequence splits and so $N \in \operatorname{add} W$. $\square$

Given a subcategory $\mathcal{X} \subseteq \operatorname{mod} \Lambda$, we say that a module $T \in \mathcal{X}$ is a *generator* for $\mathcal{X}$ if for any module $M \in \mathcal{X}$ there exists an exact sequence $0 \to N \to T' \to M \to 0$ with $T' \in \operatorname{add} T$ and $N \in \mathcal{X}$. We say that T is Ext-*projective* in $\mathcal{X}$ if $\operatorname{Ext}^i(T, M) = 0$ for any $i > 0$ and for any $M \in \mathcal{X}$. Dually, we define a *cogenerator* for $\mathcal{X}$ and an Ext-*injective* module in $\mathcal{X}$.

It follows directly from the definition that any tilting module T is an Ext-projective generator for $T^{\perp}$. Moreover $T^{\perp}$ is clearly coresolving. As a consequence of Proposition 2.2, we get the analogous properties for Wakamatsu tilting modules (see also [20, Proposition 2.6]).

2.4. Corollary. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module. Then $T^{\perp} \cap \operatorname{gen}^* T$ is a coresolving subcategory of* $\operatorname{mod} \Lambda$ *with an* Ext-*projective generator.*

Proof. $(\bigoplus_{i \in \mathbb{N}} K_i \oplus T)^{\perp}$ is coresolving and, by Proposition 2.2, it coincides with $T^{\perp} \cap \operatorname{gen}^* T$. Clearly T is a generator for $T^{\perp} \cap \operatorname{gen}^* T$ and it is projective with respect to any module in this category. $\square$

In [3, Theorems 5.4, 5.5], in order to prove that $T^{\perp}$ is covariantly finite for any tilting module T, the authors used the following result of Auslander–Buchweitz [2, Theorem 2.3, Proposition 3.6].

2.5. Proposition. *Let $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ be a coresolving subcategory with an* Ext-*projective generator and such that $\check{\mathcal{X}} = \operatorname{mod} \Lambda$. Then $\mathcal{X}$ is covariantly finite.*

Hence the correspondence stated in Theorem 1.1 can be expressed in the following way:

2.6. Proposition. *Let $T \in \operatorname{mod} \Lambda$. Then the map $T \mapsto T^{\perp}$ is a one-one correspondence between isomorphism classes of basic tilting modules and coresolving subcategories $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ with an* Ext-*projective generator such that $\check{\mathcal{X}} = \operatorname{mod} \Lambda$.*

Proof. Since $T^{\perp} \subseteq \operatorname{gen} T$ for any tilting module T, we have already observed that T is an Ext-projective generator for $T^{\perp}$. Thus the proof follows directly from Proposition 2.5 and [3, Theorem 5.5]. $\square$

We can formulate a similar partial result for Wakamatsu tilting modules.

2.7. Proposition. *Let $T \in \operatorname{mod} \Lambda$. Then $\phi : T \mapsto T^{\perp} \cap \operatorname{gen}^* T$ is an injective map between isomorphism classes of basic Wakamatsu tilting modules and coresolving subcategories $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ with an* Ext*-projective generator.*

Proof. This map exists by Corollary 2.4 and it is injective by the following lemma. □

2.8. Lemma. *Let T and T' be two basic* Ext*-projective generators of a coresolving subcategory $\mathcal{X} \subseteq \operatorname{mod} \Lambda$. Then $T \cong T'$.*

Proof. Since $T \in \mathcal{X}$ and T' is a generator for $\mathcal{X}$, there exists an exact sequence $0 \to Y \to T'_1 \to T \to 0$ where $Y \in \mathcal{X}$ and $T'_1 \in \operatorname{add} T'$. Since T is Ext-projective in $\mathcal{X}$, this sequence splits and so $T \in \operatorname{add} T'$. Similarly we get that $T' \in \operatorname{add} T$ and hence $T \cong T'$. □

In order to obtain a characterization of Wakamatsu tilting modules in terms of subcategories of $\operatorname{mod} \Lambda$, Propositions 2.3 and 2.7 show that $T^{\perp} \cap \operatorname{gen}^* T$ plays an important role. But in general we are not able to conclude that the map ϕ constructed in Proposition 2.7 is a one-one correspondence. In fact, we have the following result:

2.9. Proposition. *There exists a surjective map between coresolving subcategories $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ with an* Ext*-projective generator and isomorphism classes of basic Wakamatsu tilting modules. This map is defined by $\psi : \mathcal{X} \mapsto T$, where* $\operatorname{add} T = \mathcal{X} \cap {}^{\perp}\mathcal{X}$.

Moreover, $(\psi \circ \phi)(T) = T$ for any Wakamatsu tilting module T and $\mathcal{X} \subseteq (\phi \circ \psi)(\mathcal{X})$ for any coresolving subcategory $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ with an Ext*-projective generator.*

Proof. First let us observe that if $\mathcal{X}$ has an Ext-projective generator T, then $\mathcal{X} \cap {}^{\perp}\mathcal{X} = \operatorname{add} T$. Indeed, $\operatorname{add} T \subseteq \mathcal{X} \cap {}^{\perp}\mathcal{X}$ since T is Ext-projective in $\mathcal{X}$. Conversely, if $N \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$ there is a split exact sequence $0 \to N_1 \to T' \to N \to 0$ where $T' \in \operatorname{add} T$. Then the map ψ is well-defined.

From Proposition 2.3 it follows that ψ is surjective and that $(\psi \circ \phi)(T) = T$ for any Wakamatsu tilting module T.

To conclude, let $\mathcal{X}$ be a coresolving subcategory with an Ext-projective generator T. Then $\phi(\mathcal{X}) = T$ and $\psi(T) = T^{\perp} \cap \operatorname{gen}^* T$; moreover $\mathcal{X} \subseteq T^{\perp}$ since T is Ext-projective in $\mathcal{X}$, and $\mathcal{X} \subseteq \operatorname{gen}^* T$ since T is a generator for $\mathcal{X}$. □

Notice that the maps ϕ and ψ constructed in Propositions 2.7 and 2.9 are the same as those constructed for tilting modules in Theorem 1.1. In the original context of [3], we conclude that the maps ϕ and ψ are bijective since $(\psi \circ \phi)T = T$ for any tilting module T and $(\phi \circ \psi)\mathcal{X} = \mathcal{X}$ $(*)$ for any subcategory $\mathcal{X}$ satisfying the hypothesis of the theorem (see the proof of [3, Theorem 5.5]). In that setting, $\mathcal{X}$ is a coresolving subcategory with an Ext-projective generator such that $\check{\mathcal{X}} = \operatorname{mod} \Lambda$. The latter condition is actually essential to prove the equality $(*)$, and Corollary 2.4 shows that this is the crucial difference between tilting and Wakamatsu tilting modules. Notice also that, in the tilting case, the equality $\check{T}^{\perp} = \operatorname{mod} \Lambda$ follows from the fact that T has finite projective dimension (see [3], [1, Lemma 2.2]), which fails in general for Wakamatsu tilting modules. In particular, [20, Example 3.1] provides an example of a Wakamatsu tilting module T of infinite projective dimension such that the associated category $\mathcal{X}_T = T^{\perp} \cap \operatorname{gen}^* T = T^{\perp}$ and $\check{\mathcal{X}}_T$ is properly contained in $\operatorname{mod} \Lambda$.

Conscious of this central difference between the two contexts, we can collect the results in Propositions 2.7 and 2.9 and obtain the following correspondence:

2.10. Theorem. *Let $T \in \operatorname{mod} \Lambda$ and $\mathcal{X} \subseteq \operatorname{mod} \Lambda$. Then $\phi : T \mapsto T^{\perp} \cap \operatorname{gen}^* T$ and $\psi : \mathcal{X} \mapsto T$, where $\operatorname{add} T = \mathcal{X} \cap {}^{\perp}\mathcal{X}$, are inverse bijections between isomorphism classes of basic Wakamatsu tilting modules and coresolving subcategories with an* Ext-*projective generator, maximal among those with the same* Ext-*projective generator.*

Proof. For any coresolving subcategory with Ext-projective generator T, we have $\mathcal{X} \subseteq (\phi \circ \psi)(\mathcal{X}) = T^{\perp} \cap \operatorname{gen}^* T$. Thus, for any Wakamatsu tilting module T, $\phi(T) = T^{\perp} \cap \operatorname{gen}^* T$ is maximal among those coresolving subcategories with the same Ext-projective generator T.

Conversely, if $\mathcal{X}$ is a subcategory maximal among those with the previous properties, then $\mathcal{X} = (\phi \circ \psi)(\mathcal{X}) = T^{\perp} \cap \operatorname{gen}^* T$. □

So far we have considered Wakamatsu tilting modules as a generalization of tilting modules. In order to characterize them in terms of subcategories of $\operatorname{mod} \Lambda$, we have studied the category $T^{\perp} \cap \operatorname{gen}^* T$ as the natural analogue of the category $T^{\perp}$.

But any Wakamatsu tilting module is also Wakamatsu cotilting; thus, keeping all the notations previously defined, it is natural to consider also the category ${}^{\perp}T \cap \operatorname{cogen}^* T$ as a generalization of the category ${}^{\perp}T$. It is easy to see that for this category dual results of those obtained for $T^{\perp} \cap \operatorname{gen}^* T$ hold. Some of them are collected in the following proposition.

2.11. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module.*

(a) *For any $i \in \mathbb{N}$, let us define the modules L_i as above; then ${}^{\perp}T \cap \operatorname{cogen}^* T = {}^{\perp}(\prod_{i \in \mathbb{N}} L_i \amalg T)$.*
(b) *${}^{\perp}T \cap \operatorname{cogen}^* T$ is a resolving subcategory of $\operatorname{mod} \Lambda$ with a unique* Ext-*injective cogenerator T.*
(c) *Let $\mathcal{Y}_T = {}^{\perp}T \cap \operatorname{cogen}^* T$. Then $\mathcal{Y}_T \cap \mathcal{Y}_T{}^{\perp} = \operatorname{add} T$ and ${}^{\perp}(\mathcal{Y}_T{}^{\perp}) = \mathcal{Y}_T$.*

Thus we can easily formulate the dual version of Theorem 2.10:

2.12. Theorem. *Let $T \in \operatorname{mod} \Lambda$ and $\mathcal{Y} \subseteq \operatorname{mod} \Lambda$. Then $\phi' : T \mapsto {}^{\perp}T \cap \operatorname{cogen}^* T$ and $\psi' : \mathcal{Y} \mapsto T$, where $\operatorname{add} T = \mathcal{Y} \cap \mathcal{Y}^{\perp}$, are inverse bijections between isomorphism classes of basic Wakamatsu tilting modules and resolving subcategories with an* Ext-*injective cogenerator, maximal among those with the same* Ext-*injective cogenerator.*

Let us see how the two subcategories associated to a Wakamatsu tilting module as in Theorems 2.10 and 2.12 are related.

2.13. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module, $\mathcal{X}_T = T^{\perp} \cap \operatorname{gen}^* T$ and $\mathcal{Y}_T = {}^{\perp}T \cap \operatorname{cogen}^* T$. Then*

(a) ${}^{\perp}\mathcal{X}_T \subseteq \mathcal{Y}_T$ *and* $\mathcal{Y}_T^{\perp} \subseteq \mathcal{X}_T$.
(b) T *is an* Ext-*projective generator for* $\mathcal{Y}_T^{\perp}$ *and an* Ext-*injective cogenerator for* ${}^{\perp}\mathcal{X}_T$.

Proof. (a) Clearly $T \in \mathcal{X}_T$ and, by construction, $L_i \in \mathcal{X}_T$ for any $i \in \mathbb{N}$. By applying Proposition 2.11, we get ${}^{\perp}\mathcal{X}_T \subseteq {}^{\perp}(\prod_{i\in\mathbb{N}} L_i \amalg T) = \mathcal{Y}_T$. The other inclusion is similarly proved.

(b) Let $N \in {}^{\perp}\mathcal{X}_T$ and let I be its injective envelope. Since $I \in \mathcal{X}_T$, there exists a commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d, hook] & N \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & L \arrow[r] & T_0 \arrow[r] & I \arrow[r] & 0 \end{tikzcd}$$

where $T_0 \in \operatorname{add} T$ and $L \in \mathcal{X}_T$. Since $\operatorname{Ext}^1({}^{\perp}\mathcal{X}_T, \mathcal{X}_T) = 0$, the first row splits and so N is cogenerated by T. Take now a left $\operatorname{add} T$-approximation of N, $0 \to N \to T^0 \to N_1 \to 0$, where $T^0 \in \operatorname{add} T$ and $N_1 \in {}^{\perp}T$. Let $X \in \mathcal{X}_T$. Since there exists an exact sequence $0 \to X_1 \to T_1 \to X \to 0$ with $T_1 \in \operatorname{add} T$ and $X_1 \in \mathcal{X}_T$, $\operatorname{Ext}^1(N_1, X) \cong \operatorname{Ext}^2(N_1, X_1) \cong \operatorname{Ext}^1(N, X_1) = 0$. Similarly $\operatorname{Ext}^i(N_1, X) = 0$ for any $i > 0$ and any $X \in \mathcal{X}_T$. Thus T is an Ext-injective cogenerator for ${}^{\perp}\mathcal{X}_T$; similarly we can prove that T is an Ext-projective generator for $\mathcal{Y}_T^{\perp}$. $\square$

The following example shows that in general the inclusions in Proposition 2.13(a) are strict.

2.14. Example. Let $\Lambda = k[x]/(x^2)$ for a field k. Then Λ is a tilting module, so in particular it is a Wakamatsu tilting module. In this case the category ${}^{\perp}\mathcal{X}_\Lambda$ coincides with $\operatorname{add} \Lambda$. Since Λ is injective, the simple module S belongs to $\mathcal{Y}_\Lambda = \operatorname{cogen}^* \Lambda = \operatorname{mod} \Lambda$ but not to ${}^{\perp}\mathcal{X}_\Lambda$.

2.15. Remark. Proposition 2.13 and Example 2.14 show that the maximality of the categories required in Theorems 2.10 and 2.12 cannot be dropped. Indeed, for any Wakamatsu tilting module T, the subcategories $\mathcal{Y}_T$ and ${}^{\perp}\mathcal{X}_T$ (and dually $\mathcal{X}_T$ and $\mathcal{Y}_T^{\perp}$) are both resolving subcategories, with the same Ext-injective cogenerator, but in general ${}^{\perp}\mathcal{X}_T \subsetneq \mathcal{Y}_T$. So $\psi'(\mathcal{Y}_T) = \psi'({}^{\perp}\mathcal{X}_T)$, but ${}^{\perp}\mathcal{X}_T \subseteq (\phi' \circ \psi')({}^{\perp}\mathcal{X}_T) = \mathcal{Y}_T$.

Notice also that, in Example 2.14, both the category $\mathcal{Y}_\Lambda$ and the category ${}^{\perp}\mathcal{X}_\Lambda$ are functorially finite. Thus, in Theorems 2.10 and 2.12, we couldn't avoid the condition of maximality on the subcategories corresponding to Wakamatsu tilting modules, even if we would assume covariant or contravariant finiteness.

3. Cotorsion theories

For $\mathcal{A} = \operatorname{mod} \Lambda$ or $\mathcal{A} = \operatorname{Mod} \Lambda$, a pair $(\mathcal{C}, \mathcal{D})$ of subcategories of $\mathcal{A}$ is called a *cotorsion theory* if $\mathcal{C} = \{M \in \mathcal{A}\colon \operatorname{Ext}^1(M, \mathcal{D}) = 0\}$ and $\mathcal{D} = \{N \in \mathcal{A}\colon \operatorname{Ext}^1(\mathcal{C}, N) = 0\}$. The class $\omega = \mathcal{C} \cap \mathcal{D}$ is called *the kernel* of the cotorsion theory. A cotorsion theory $(\mathcal{C}, \mathcal{D})$ in $\mathcal{A}$ is called *complete* if $\mathcal{C}$ is contravariantly finite and $\mathcal{D}$ is covariantly finite (see [18]). In this section we investigate the relationship between Wakamatsu tilting modules and cotorsion pairs.

Notice that, in a cotorsion theory $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ is resolving and $\mathcal{D}$ is coresolving, it follows by dimension shift that $\mathcal{C} = {}^{\perp}\mathcal{D}$ and $\mathcal{D} = \mathcal{C}^{\perp}$ (see [3, Lemma 3.1]). We assume that all the cotorsion theories considered in this section are of this type.

When T is a tilting module, we know from [3] that the naturally associated pair $(\mathcal{C}, \mathcal{D})$ of subcategories, where $\mathcal{D} = T^{\perp}$ and $\mathcal{C} = \{M \in \operatorname{mod} \Lambda\colon \operatorname{Ext}^1(M, \mathcal{D})\}$, is a complete cotorsion pair in $\operatorname{mod} \Lambda$.

In Theorems 2.10 and 2.12 we showed that we can associate to any Wakamatsu tilting module a pair of subcategories of $\operatorname{mod} \Lambda$. In the present section we will prove that these subcategories give rise to a pair of cotorsion theories in $\operatorname{mod} \Lambda$. Hence we formulate the results of the previous section in terms of cotorsion theories.

For any algebra Λ, we can define two partial orders on the cotorsion theories in $\operatorname{mod} \Lambda$ or in $\operatorname{Mod} \Lambda$: by inclusion on their first component or on their second component. In the sequel, given two cotorsion theories $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}', \mathcal{D}')$ in $\mathcal{A}$, we will write $(\mathcal{C}, \mathcal{D}) \leqslant (\mathcal{C}', \mathcal{D}')$ if $\mathcal{C} \subseteq \mathcal{C}'$.

Let T be a Wakamatsu tilting module and let $\mathcal{X}_T$ and $\mathcal{Y}_T$ be the subcategories of $\operatorname{mod} \Lambda$ defined in the previous section. The next result follows directly from Propositions 2.3, 2.11, and 2.13.

3.1. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module. Then $({}^{\perp}\mathcal{X}_T, \mathcal{X}_T)$ and $(\mathcal{Y}_T, \mathcal{Y}_T^{\perp})$ are two cotorsion theories in $\operatorname{mod} \Lambda$ with the same kernel $\operatorname{add} T$ and such that $({}^{\perp}\mathcal{X}_T, \mathcal{X}_T) \leqslant (\mathcal{Y}_T, \mathcal{Y}_T^{\perp})$.*

Hence, we can associate to any Wakamatsu tilting module a pair of cotorsion theories in $\operatorname{mod} \Lambda$. We cannot say anything in general about their completeness.

Conversely, it is interesting to see that a cotorsion theory with some additional assumptions determines a Wakamatsu tilting module.

3.2. Proposition. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion theory in $\operatorname{mod} \Lambda$. The following conditions are equivalent*:

(a) *there exists an* Ext-*injective cogenerator for $\mathcal{C}$*;
(b) *there exists an* Ext-*projective generator for $\mathcal{D}$*.

If (a) *is satisfied, then $\mathcal{C} \cap \mathcal{D} = \operatorname{add} T$ for a Wakamatsu tilting module T.*

Proof. (b) $\Rightarrow$ (a). From Proposition 2.9, it follows that $\mathcal{D} \cap \mathcal{C} = \operatorname{add} T$ where T is the Ext-projective generator for $\mathcal{D}$. Then T is a Wakamatsu tilting module and $\mathcal{D} \subseteq T^{\perp} \cap \operatorname{gen}^* T$.

By an argument similar to the one used in the proof of Proposition 2.13(b), it follows that T is an Ext-injective cogenerator for ${}^{\perp}\mathcal{D} = \mathcal{C}$. The other implication is proved similarly. $\square$

3.3. Corollary. *Let* $(\mathcal{C}, \mathcal{D})$ *be a cotorsion theory in* $\operatorname{mod}\Lambda$ *satisfying the conditions in* 3.2. *Then* $({}^{\perp}\mathcal{X}_T, \mathcal{X}_T) \leqslant (\mathcal{C}, \mathcal{D}) \leqslant (\mathcal{Y}_T, \mathcal{Y}_T^{\perp})$ *for a Wakamatsu tilting module* T.

Proof. From Proposition 3.2, we get that $\mathcal{C} \cap \mathcal{D} = \operatorname{add} T$, where T is a Wakamatsu tilting module. Since T is an Ext-injective cogenerator for $\mathcal{C}$, then $\mathcal{C} \subseteq {}^{\perp}T \cap \operatorname{cogen}^* T = \mathcal{Y}_T$. The module T is also an Ext-projective generator for $\mathcal{D}$, so $\mathcal{D} \subseteq T^{\perp} \cap \operatorname{gen}^* T = \mathcal{X}_T$. $\square$

Thus we can express the results stated in Theorems 2.10 and 2.12 in the language of cotorsion theories.

3.4. Theorem. *Let* $T \in \operatorname{mod}\Lambda$ *and let* $(\mathcal{C}, \mathcal{D})$ *be a cotorsion theory in* $\operatorname{mod}\Lambda$. *Then*:

(a) $T \mapsto ({}^{\perp}\mathcal{X}_T, \mathcal{X}_T)$ *is a one-one correspondence between isomorphism classes of basic Wakamatsu tilting modules and cotorsion theories minimal among those with the same kernel and the first category having an* Ext-*injective cogenerator. The inverse correspondence is given by* $(\mathcal{C}, \mathcal{D}) \mapsto T$, *where* $\operatorname{add} T = \mathcal{C} \cap \mathcal{D}$.

(b) $T \mapsto (\mathcal{Y}_T, \mathcal{Y}_T^{\perp})$ *is a one-one correspondence between isomorphism classes of basic Wakamatsu tilting modules and cotorsion theories maximal among those with the same kernel and the first category having an* Ext-*injective cogenerator. The inverse correspondence is given by* $(\mathcal{C}, \mathcal{D}) \mapsto T$, *where* $\operatorname{add} T = \mathcal{C} \cap \mathcal{D}$.

Studying Wakamatsu tilting modules by means of cotorsion theories, we can recognize better how the two correspondences stated in Theorems 2.10 and 2.12 are related. Indeed, they are actually the minimal or the maximal version of Theorem 3.4.

Our next goal is to extend the cotorsion theories in $\operatorname{mod}\Lambda$, associated to a Wakamatsu tilting module $T \in \operatorname{mod}\Lambda$, to a pair of cotorsion theories in $\operatorname{Mod}\Lambda$. We will show that this can be done very naturally, in such a way that the properties in Proposition 3.2 still hold. Moreover, the cotorsion theories obtained with this process turn out to be complete. Notice that cotorsion theories in $\operatorname{Mod}\Lambda$ associated with possibly infinitely generated cotilting modules have been investigated in [15].

First we need to introduce some new notations. For the rest of this section, given a module $M \in \operatorname{Mod}\Lambda$ or a subcategory $\mathcal{C} \subseteq \operatorname{Mod}\Lambda$, we will denote by ${}^{\perp}M$ or $M^{\perp}$, ${}^{\perp}\mathcal{C}$ or $\mathcal{C}^{\perp}$ the left or right orthogonal of M or $\mathcal{C}$ in $\operatorname{Mod}\Lambda$. Given a module $M \in \operatorname{Mod}\Lambda$, we denote by $\operatorname{Gen}^* M$ the class of modules $N \in \operatorname{Mod}\Lambda$ such that there exists an exact sequence $\cdots \xrightarrow{f_2} M^1 \xrightarrow{f_1} M^0 \xrightarrow{f_0} N \to 0$ with $M^i \in \operatorname{Add} M$ and $\operatorname{Ext}^1(M, \ker f_i) = 0$ for any $i \in \mathbb{N}$. Dually, we define the class $\operatorname{Cogen}^* M$ as the class of modules $N \in \operatorname{Mod}\Lambda$ such that there exists an exact sequence $0 \to N \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \cdots$, with $M_j \in \operatorname{Prod} M$ and $\operatorname{Ext}^1(\operatorname{coker} g_j, M) = 0$ for any $j \in \mathbb{N}$.

3.5. Remark. Following the definition in [14], given an arbitrary associative ring R, a module $M \in \operatorname{Mod} R$ is called *product complete* if every product of copies of M is a direct summand of a coproduct of copies of M. It follows that $M \in \operatorname{Mod} R$ is product complete if and only if $\operatorname{Add} M = \operatorname{Prod} M$. Indeed, if M is product complete, we know by definition that $\operatorname{Prod} M \subseteq \operatorname{Add} M$. Moreover, by [14, Proposition 3.7], $M^{(I)}$ is pure-injective for any cardinal I. Thus the pure exact sequence $0 \to M^{(I)} \to M^I \to X \to 0$ splits, and so $\operatorname{Add} M \subseteq \operatorname{Prod} M$.

As is well known, any finitely generated module over an Artin algebra Λ is product complete (see [14, Proposition 3.9]). It follows that, if $M \in \operatorname{mod} \Lambda$ and $N \in \operatorname{mod} \Lambda$, for any cardinal α there exist cardinals β and γ such that

$$\operatorname{Ext}^i\big(M^\alpha, N\big) \overset{\oplus}{\leqslant} \operatorname{Ext}^i(M, N)^\beta \quad \text{and} \quad \operatorname{Ext}^j\big(N, M^{(\alpha)}\big) \overset{\oplus}{\leqslant} \operatorname{Ext}^j(N, M)^\gamma$$

for any $i, j \in \mathbb{N}$.

Given a Wakamatsu tilting module $T \in \operatorname{mod} \Lambda$, we want to describe the subcategories of $\operatorname{Mod} \Lambda$ which naturally generalize those studied in the previous section: $T^\perp \cap \operatorname{Gen}^* T$ and ${}^\perp T \cap \operatorname{Cogen}^* T$. We first show that the same characterizations as those proved in Propositions 2.2 and 2.11 still hold.

3.6. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module. Then*:

(a) $T^\perp \cap \operatorname{Gen}^* T = (\bigoplus_{i \in N} K_i \oplus T)^\perp$.
(b) ${}^\perp T \cap \operatorname{Cogen}^* T = {}^\perp(\prod_{i \in \mathbb{N}} L_i \,\Pi\, T)$.

Proof. The proof of (a) is similar to that of Proposition 2.2, since T and K_i are product complete for any $i \geqslant 0$; hence $\operatorname{Ext}^j(K_i^\alpha, T^{(\beta)}) = 0$ for any $i \geqslant 0$ and for any $j \geqslant 1$. Part (b) follows dually. $\square$

3.7. Remark. Let $M \in \operatorname{Mod} \Lambda$ and let $\Omega^i(M)$ be the i-syzygy of M. Then $M^\perp = \{N \in \operatorname{Mod} \Lambda\colon \operatorname{Ext}^1(\bigoplus_{i \in \mathbb{N}} \Omega^i(M) \oplus M, N) = 0\}$, as it can be easily seen by dimension shift.

In the sequel, in analogy with the notations used in $\operatorname{mod} \Lambda$, we will denote by $\overline{\mathcal{X}}_T$ and $\overline{\mathcal{Y}}_T$ the subcategories $T^\perp \cap \operatorname{Gen}^* T$ and ${}^\perp T \cap \operatorname{Cogen}^* T$ of $\operatorname{Mod} \Lambda$. Moreover, following [8] a class of modules in $\operatorname{Mod} \Lambda$ is called *definable* if it is closed under products, direct limits, and pure submodules.

3.8. Theorem. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module. Then*:

(a) $({}^\perp\overline{\mathcal{X}}_T, \overline{\mathcal{X}}_T)$ *and* $(\overline{\mathcal{Y}}_T, \overline{\mathcal{Y}}_T^\perp)$ *are two complete cotorsion theories in* $\operatorname{Mod} \Lambda$ *with kernel* $\operatorname{Add} T$ *and such that* $({}^\perp\overline{\mathcal{X}}_T, \overline{\mathcal{X}}_T) \leqslant (\overline{\mathcal{Y}}_T, \overline{\mathcal{Y}}_T^\perp)$.
(b) *The objects in* $\operatorname{Mod} \Lambda$ *have* minimal *right* $\overline{\mathcal{Y}}_T$*-approximations and* minimal *left* $\overline{\mathcal{Y}}_T^\perp$*-approximations.*
(c) $\overline{\mathcal{X}}_T$ *is a definable class.*

Proof. (a) Since T is product complete, the proofs of the similar results in $\operatorname{mod}\Lambda$ still work. Moreover, from Proposition 3.6 and Remark 3.7, we get that $({}^{\perp}\overline{\mathcal{X}}_T, \overline{\mathcal{X}}_T)$ is the cotorsion theory cogenerated by the module $\bigoplus_{i\in\mathbb{N}} K_i \oplus T$, that is

$$\overline{\mathcal{X}}_T = \left(\bigoplus_{i\in\mathbb{N}} K_i \oplus T\right)^{\perp}.$$

Dually, $(\overline{\mathcal{Y}}_T, \overline{\mathcal{Y}}_T^{\perp})$ is the cotorsion theory generated by the module $\prod_{i\in\mathbb{N}} L_i \ \Pi\ T$, that is $\overline{\mathcal{Y}}_T = {}^{\perp}(\prod_{i\in\mathbb{N}} L_i \ \Pi\ T)$. Notice that, since T and L_i are finitely generated and so pure-injective, the module $\prod_{i\in\mathbb{N}} L_i \ \Pi\ T$ is pure-injective as well. Thus, by [18, Theorems 2.3, 2.8], both cotorsion theories are complete.

(b) As we have already noticed, the module $\prod_{i\in\mathbb{N}} L_i \ \Pi\ T$ is pure-injective. Thus the functors $\operatorname{Ext}^j(-, \prod_{i\in\mathbb{N}} L_i \ \Pi\ T)$ commute with direct limits for any $j \geqslant 0$; in particular $\mathcal{Y}_T$ is closed under direct limits. It follows from [18, Corollary 1.19] that any module in $\operatorname{Mod}\Lambda$ admits minimal right $\mathcal{Y}_T$-approximations and minimal left $\mathcal{Y}_T^{\perp}$-approximations.

(c) For any finitely presented module N, the functor $\operatorname{Ext}^1(N, -)$ is coherent. Since T, K_i and all their syzygies are finitely presented, $\mathcal{X}_T$ is the intersection of the kernels of a family of coherent functors. This is equivalent to saying that $\mathcal{X}_T$ is a definable class (see [8, Section 2.3]). □

The previous proposition shows that the cotorsion theories $({}^{\perp}\mathcal{X}_T, \mathcal{X}_T)$ and $(\mathcal{Y}_T, \mathcal{Y}_T^{\perp})$ in $\operatorname{mod}\Lambda$, associated to a Wakamatsu tilting module T, can be naturally extended to a pair of cotorsion theories in $\operatorname{Mod}\Lambda$. Let us prove that the extended cotorsion theories have the same properties as those stated in Proposition 3.2.

3.9. Proposition. *Let $T \in \operatorname{mod}\Lambda$ be a Wakamatsu tilting module. Then*:

(a) *T is an* Ext-*injective cogenerator for $\overline{\mathcal{Y}}_T$ and ${}^{\perp}\overline{\mathcal{X}}_T$.*
(b) *T is an* Ext-*projective generator for $\overline{\mathcal{X}}_T$ and $\overline{\mathcal{Y}}_T^{\perp}$.*

Proof. Let $N \in \overline{\mathcal{Y}}_T$. Since $T \in \overline{\mathcal{Y}}_T^{\perp}$, T is Ext-injective in $\overline{\mathcal{Y}}_T$. Since the cotorsion theory $(\overline{\mathcal{Y}}_T, \overline{\mathcal{Y}}_T^{\perp})$ is complete, any module in $\operatorname{Mod}\Lambda$ admits a (minimal) left $\overline{\mathcal{Y}}_T^{\perp}$-approximation. Thus there exists an exact sequence $0 \to N \to C \to N_1 \to 0$ where $C \in \overline{\mathcal{Y}}_T^{\perp}$ and $N_1 \in {}^{\perp}(\overline{\mathcal{Y}}_T^{\perp}) = \overline{\mathcal{Y}}_T$. So $C \in \overline{\mathcal{Y}}_T \cap \overline{\mathcal{Y}}_T^{\perp} = \operatorname{Add} T = \operatorname{Prod} T$; we conclude that T is an Ext-injective cogenerator for $\overline{\mathcal{Y}}_T$. The other statements are proved similarly. □

Notice that, moving from $\operatorname{mod}\Lambda$ to $\operatorname{Mod}\Lambda$, the complete symmetry between the two cotorsion theories associated to a Wakamatsu tilting module is broken. Indeed, they have similar properties but $\overline{\mathcal{Y}}_T$ provides *minimal* left approximations, while in general we don't know whether the dual property holds also for ${}^{\perp}\overline{\mathcal{X}}_T$. Moreover $\overline{\mathcal{X}}_T$ is a definable class but we cannot say anything about $\overline{\mathcal{Y}}_T^{\perp}$. On the other hand, while in $\operatorname{mod}\Lambda$ we cannot say anything about the covariant or contravariant finiteness of the classes, in $\operatorname{Mod}\Lambda$ we obtain the completeness of both cotorsion theories.

4. Wakamatsu tilting modules of finite projective dimension

It is interesting to investigate the relationship between tilting modules and Wakamatsu tilting modules. There are Wakamatsu tilting Λ-modules T with $\mathrm{pd}_\Lambda T = \infty$. If T is a Wakamatsu tilting module with $\mathrm{pd}_\Lambda T$ and $\mathrm{pd}_\Gamma T$ finite, where $\Gamma = \mathrm{End}({}_\Lambda T)^{\mathrm{op}}$, then T is a tilting module. It is however an open problem whether a Wakamatsu tilting module of finite projective dimension must be a tilting module. This is known as Wakamatsu tilting conjecture (see [5, Chapter III]), and it is interesting also because of a close relationship to well known homological conjectures. This motivates investigating Wakamatsu tilting modules of finite projective dimension more closely. In the present section we generalize some results on tilting modules to this setting, and point out further relationships to the homological conjectures.

A module $M \in \mathrm{mod}\,\Lambda$ is *partial tilting* if it is a direct summand of a tilting module. Any module $X \in \mathrm{mod}\,\Lambda$ such that $M \oplus X$ is a tilting module and $\mathrm{add}\,M \cap \mathrm{add}\,X = 0$ is a *complement* of M. A module $M \in \mathrm{mod}\,\Lambda$ is *almost complete tilting* if it is partial tilting and $\delta(M) = \delta(\Lambda) - 1$. In [6,12] the authors formulated and studied the following conjecture involving almost complete tilting modules: *Any almost complete tilting module admits a finite number of non-isomorphic indecomposable complements.* It was proved in [6,12] that the latter conjecture restricted to projective modules is equivalent to the *Generalized Nakayama Conjecture*, which says that any indecomposable injective Λ-module is a summand of an injective module appearing in a minimal injective resolution of Λ (see [3,11]). Moreover, as was pointed out by Buan, the latter conjecture is a consequence of the Wakamatsu tilting conjecture. After giving some results on Wakamatsu tilting modules of finite projective dimension, we shall give more information on the relationship between the conjectures.

For a Wakamatsu tilting module T we have seen that the classes ${}^\perp\mathcal{X}_T$ and $\mathcal{Y}_T$, where $\mathcal{X}_T = T^\perp \cap \mathrm{gen}^* T$ and $\mathcal{Y}_T = {}^\perp T \cap \mathrm{cogen}^* T$, are different in general. If however $\mathrm{pd}_\Lambda T$ is finite, we have equalities for some related subcategories.

4.1. Proposition. *Let $T \in \mathrm{mod}\,\Lambda$ be a Wakamatsu tilting module with $\mathrm{pd}_\Lambda T < \infty$. Then ${}^\perp T \cap \mathcal{P}^{<\infty} = \mathcal{Y}_T \cap \mathcal{P}^{<\infty} = {}^\perp\mathcal{X}_T \cap \mathcal{P}^{<\infty}$.*

Proof. First, let us prove that ${}^\perp T \cap \mathcal{P}^{<\infty} = \mathcal{Y}_T \cap \mathcal{P}^{<\infty}$. One inclusion is trivial, since $\mathcal{Y}_T \subseteq {}^\perp T$.

So, let $C \in {}^\perp T \cap \mathcal{P}^{<\infty}$ and let I be the injective envelope of C. Since $I \in \mathcal{X}_T$, there exists an exact sequence $\cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} I \to 0$ where $T_i \in \mathrm{add}\,T$. Thus we construct the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & T_1 \arrow[r, "f_1"] & T_0 \arrow[r, "f_0"] & I \arrow[r] & 0
\end{tikzcd}$$

where $K_0 = \ker f_0$. We want to prove that $\mathrm{Ext}^1(C, K_0) = 0$, that is the first row splits. Writing $K_i = \ker f_i$ and by applying the functor $\mathrm{Hom}(C, -)$ to the long exact sequence $\cdots \to T_2 \xrightarrow{f_2} T_1 \xrightarrow{f_1} K_0 \to 0$, since $C \in {}^{\perp}T$ we obtain the isomorphisms

$$\mathrm{Ext}^1(C, K_0) \cong \mathrm{Ext}^2(C, K_1) \cong \cdots \cong \mathrm{Ext}^{n+1}(C, K_n)$$

for any $n \in \mathbb{N}$. Since $C \in \mathcal{P}^{<\infty}$, we conclude that $\mathrm{Ext}^1(C, K_0) = 0$. Thus, looking at the previous diagram, we infer that there is an embedding $0 \to C \to T_0$. So let us consider a minimal left $\mathrm{add}\, T$-approximation $0 \to C \xrightarrow{g_0} T_0' \to \operatorname{coker} g_0 \to 0$ of C, where $T_0' \in \mathrm{add}\, T$. Notice that $\mathrm{Ext}^1(\operatorname{coker} g_0, T) = 0$ by Wakamatsu's lemma and $\mathrm{Ext}^i(\operatorname{coker} g_0, T) = 0$ for any $i > 1$ since $C, T \in {}^{\perp}T$. Moreover $\operatorname{coker} g_0$ belongs to ${}^{\perp}T \cap \mathcal{P}^{<\infty}$, since both C and T have finite projective dimension. Repeating the same argument for $\operatorname{coker} g_0$, we obtain a long exact sequence $0 \to C \xrightarrow{g_0} T_0 \xrightarrow{g_1} T_1 \to \cdots$, where $T_i \in \mathrm{add}\, T$ and $\operatorname{coker} g_i \in {}^{\perp}T$. Then $C \in \mathcal{Y}_T \cap \mathcal{P}^{<\infty}$.

Let now $N \in {}^{\perp}T \cap \mathcal{P}^{<\infty}$ and $M \in \mathcal{X}_T$. Since there exists an exact sequence $\cdots \to T_2 \to T_1 \to T_0 \to M \to 0$ with $T_i \in \mathrm{add}\, T$, by dimension shift we obtain $\mathrm{Ext}^i(N, M) = 0$ for any $i > 0$. Thus ${}^{\perp}T \cap \mathcal{P}^{<\infty} \subseteq {}^{\perp}\mathcal{X}_T \cap \mathcal{P}^{<\infty} \subseteq \mathcal{Y}_T \cap \mathcal{P}^{<\infty}$, where the latter inclusion follows by Proposition 2.13. From the first part of the proof, it follows that all these inclusions are actually equalities. □

4.2. Remark. Given a tilting module $T \in \mathrm{mod}\, \Lambda$, we infer from [3] and [13] that the categories ${}^{\perp}T \cap \mathcal{P}^{<\infty}$ and $\check{\mathrm{add}}\, T$ coincide, where the latter category is clearly contained in $\mathcal{Y}_T \cap \mathcal{P}^{<\infty}$. Thus, Proposition 4.1 generalizes this property of tilting modules to Wakamatsu tilting modules of finite projective dimension. It also proves that the equality $\mathcal{Y}_T \cap \mathcal{P}^{<\infty} = \check{\mathrm{add}}\, T$ holds for any tilting module T.

We can now give some equivalent formulations for a Wakamatsu tilting module to be a tilting module.

4.3. Proposition. *Let $T \in \mathrm{mod}\, \Lambda$ be a Wakamatsu tilting module with $\mathrm{pd}_{\Lambda} T < \infty$. The following conditions are equivalent*:

(i) *T is a tilting module.*
(ii) *${}^{\perp}T \cap \mathcal{P}^{<\infty}$ is contravariantly finite.*
(iii) $\sup\{\mathrm{pd}\, M \mid M \in {}^{\perp}T \cap \mathcal{P}^{<\infty}\} = n < \infty$.

Proof. (i) ⇒ (ii). If T is a tilting module, then ${}^{\perp}T \cap \mathcal{P}^{<\infty}$ is contravariantly finite by [13, Theorem 2.1].

(ii) ⇒ (iii). Since ${}^{\perp}T \cap \mathcal{P}^{<\infty}$ is a contravariantly finite and resolving subcategory contained in $\mathcal{P}^{<\infty}$, it follows from [3, Corollary 3.9] that $\sup\{\mathrm{pd}\, X \mid X \in {}^{\perp}T \cap \mathcal{P}^{<\infty}\} = n < \infty$.

(iii) $\Rightarrow$ (i). Let us suppose that $\sup\{\operatorname{pd} M \mid M \in {}^{\perp}T \cap \mathcal{P}^{<\infty}\} = n < \infty$. Let us consider a long exact sequence $0 \to \Lambda \xrightarrow{g_0} T_0 \xrightarrow{g_1} T_1 \xrightarrow{g_2} T_2 \to \cdots$ where $T_i \in \operatorname{add} T$ and $K_i = \operatorname{coker} g_i \in {}^{\perp}T$. For any $m \in \mathbb{N}$ the following isomorphisms hold:

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^2(K_m, K_{m-2}) \cong \cdots \cong \operatorname{Ext}^{m+1}(K_m, \Lambda).$$

Since $K_n \in \mathcal{P}^{<\infty}$, by hypothesis we know that $\operatorname{Ext}^{n+1}(K_n, -) = 0$; it follows that $\operatorname{Ext}^1(K_n, K_{n-1}) = 0$ and so $K_n \in \operatorname{add} T$. Thus there exists a finite T-coresolution of Λ, and hence T is a tilting module. $\square$

Thanks to Proposition 4.3, we can tell whether a Wakamatsu tilting module is actually tilting by considering the subcategory ${}^{\perp}T \cap \mathcal{P}^{<\infty}$. We have the following consequences.

4.4. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module with $\operatorname{pd}_\Lambda T < \infty$. Then*:

(i) *If the finitistic dimension of Λ is finite, then T is a tilting module.*
(ii) *If $\operatorname{id}_\Lambda \Lambda < \infty$, then T is a tilting module.*

Proof. (i) follows directly from 4.3.

(ii) This is proved also in [5, Chapter III, Proposition 3.8], but we include a direct proof here. If $\operatorname{id}_\Lambda \Lambda = n < \infty$, we can argue as in the proof of Proposition 4.3. So, let us consider a long exact sequence $0 \to \Lambda \xrightarrow{g_0} T_0 \xrightarrow{g_1} T_1 \xrightarrow{g_2} T_2 \to \cdots$ where $T_i \in \operatorname{add} T$ and $K_i = \operatorname{coker} g_i \in {}^{\perp}T$. Then for any $n \in \mathbb{N}$ we obtain

$$\operatorname{Ext}^1(K_n, K_{n-1}) \cong \operatorname{Ext}^2(K_n, K_{n-2}) \cong \cdots \cong \operatorname{Ext}^{n+1}(K_n, \Lambda).$$

As $\operatorname{Ext}^{n+1}(-, \Lambda) = 0$, we get that $\operatorname{Ext}^1(K_n, K_{n-1}) = 0$ and so $K_n \in \operatorname{add} T$. Thus T is a tilting module, since there exists a finite T-coresolution of Λ. $\square$

As a final remark, let us consider the following result of Buan and Solberg, proved in [6, Theorem 3.6].

4.5. Theorem. *Let $M \in \operatorname{mod} \Lambda$ be an almost complete tilting module. Then M has an infinite number of non-isomorphic indecomposable complements if and only if M is a Wakamatsu tilting module.*

It follows that the Wakamatsu tilting conjecture (WTC) implies that any almost complete tilting module has only a finite number of complements, which again implies the Generalized Nakayama Conjecture, as we have already pointed out. In fact, this is true also if (WTC) is replaced by the following conjecture: Any Wakamatsu partial tilting module of finite projective dimension is a tilting module. This motivates investigating Wakamatsu partial tilting modules of finite projective dimension, as we will do in the next section. Notice also that it is a special case of Theorem 4.5 that an almost complete Wakamatsu tilting module has an infinite number of non-isomorphic complements. We shall give a generalization of this in the next section.

5. Complements to Wakamatsu tilting modules

Motivated by the results of the previous section we shall investigate Wakamatsu tilting partial tilting modules which are not tilting modules. We get results about the complements which are similar to known results for almost complete tilting modules. In particular we show that a Wakamatsu tilting module which is a partial tilting module, but not a tilting module, has an infinite number of non-isomorphic complements.

In [7,11,12] the authors proved that all the indecomposable complements of an almost complete tilting module $M \in \operatorname{mod} \Lambda$ occur as cokernels of a long exact sequence whose terms are modules in $\operatorname{add} M$. Following some ideas contained in the quoted papers, we generalize the latter result to Wakamatsu tilting modules. In particular, given a Wakamatsu tilting module T, we construct a long exact sequence in which the cokernels of the maps are complements of T.

The next lemma is dual to [7, Proposition 1.3]. For a complete proof of it, see [16].

5.1. Lemma. *Let $M \in \operatorname{mod} \Lambda$ be a partial tilting module and let X be a complement of M. Let $T = M \oplus Y$ be an exceptional module. Suppose that there exists an exact sequence*

$$0 \to X \to T' \to T'' \to 0$$

with $T', T'' \in \operatorname{add} T$. Then T is a tilting module.

Let now T be both a Wakamatsu tilting module of finite projective dimension and a partial tilting module. Then we obtain the following result, analogous to [7, Proposition 1.3(2)].

5.2. Lemma. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module of finite projective dimension. If M is a complement of T, then there exists at least another complement of T non-isomorphic to M.*

Proof. Since M is a complement of T, then $M \in {}^{\perp}T \cap \mathcal{P}^{<\infty}$ and so, by Proposition 4.1, M is cogenerated by T. Thus there exists an exact sequence $0 \to M \xrightarrow{i} T_0 \to K \to 0$, where $T_0 \in \operatorname{add} T$ and i is a minimal left $\operatorname{add} T$-approximation of M. In particular $\operatorname{Ext}^1(K, T) = 0$ and, since $M, T \in {}^{\perp}T$, then $\operatorname{Ext}^i(K, T) = 0$ for any $i > 0$. By applying the functors $\operatorname{Hom}(T, -)$, $\operatorname{Hom}(-, M)$ and $\operatorname{Hom}(K, -)$ to the previous exact sequence, we get $\operatorname{Ext}^i(T, K) = 0$ and $\operatorname{Ext}^i(K, K) \cong \operatorname{Ext}^{i+1}(K, M) \cong \operatorname{Ext}^i(M, M) = 0$ for any $i > 0$. Then also $T \oplus K$ is a selforthogonal module of finite projective dimension. It follows from Lemma 5.1 that $T \oplus K$ is a tilting module. Moreover $\operatorname{Ext}^i(K, K) = 0$ but $\operatorname{Ext}^1(K, M) \neq 0$, and so K is not isomorphic to M. To conclude, let us prove that $\operatorname{add} K \cap \operatorname{add} T = 0$. If $K = K_1 \oplus T'$ with $T' \in \operatorname{add} T$, we could construct the following commutative pullback diagram:

$$\begin{tikzcd}
& & 0 & 0 & \\
& & T_1 \arrow[u] \arrow[r, equal] & T_1 \arrow[u] & \\
0 \arrow[r] & M \arrow[r, "i"] & T_0 \arrow[r] \arrow[u] & K_1 \oplus T_1 \arrow[r] \arrow[u, "\pi"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & K_1 \arrow[r] \arrow[u, "j"] & 0 \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $M, K_1 \in T^{\perp}$, also Q belongs to $T^{\perp}$ and so the second column splits. Thus i is not a minimal left $\operatorname{add} T$-approximation of M. $\square$

By applying repeatedly Lemma 5.2, whatever value $\delta(T)$ assumes, we can construct a long exact sequence whose cokernels are non-isomorphic complements of T.

5.3. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a Wakamatsu tilting module which is a partial tilting module but not a tilting module. If X_0 is a complement of T, then*:

(a) *T admits infinitely many non-isomorphic complements.*
(b) *There exists a long exact sequence*

$$0 \to X_0 \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots$$

where $T_i \in \operatorname{add} T$ and $\operatorname{coker} f_i = X_{i+1}$ for $i \geqslant 0$, such that $\{X_i\}_{i \geqslant 0}$ is a family of non-isomorphic complements of T. In addition, each $0 \to X_i \to T_{i+1}$ is a minimal left $\operatorname{add} T$-approximation and each $T_j \to X_j \to 0$ is a minimal right $\operatorname{add} T$-approximation.

Proof. By Lemma 5.2 there exists a complement X_1 not isomorphic to X_0, constructed as the cokernel of a minimal left $\operatorname{add} T$-approximation of X_0. We can repeat the same argument for X_1 in order to obtain another complement X_2 non-isomorphic to X_1. Going on in this way, we obtain a family $\{X_n\}_{n \in \mathbb{N}}$ of complements where $X_i \ncong X_{i-1}$. Notice that, by construction, $\operatorname{Ext}^1(X_n, X_{n-1}) \neq 0$ and $\operatorname{Ext}^1(X_n, X_{n-1}) \cong \operatorname{Ext}^2(X_n, X_{n-2}) \cong \cdots \cong \operatorname{Ext}^j(X_n, X_{n-j})$ for any $n \in \mathbb{N}$ and $j \geqslant 1$. It follows that for, a fixed $\bar{n}$, $X_{\bar{n}} \ncong X_i$ for any $i < \bar{n}$: indeed, if there exists $j < \bar{n}$ such that $X_j \cong X_{\bar{n}}$, then $0 = \operatorname{Ext}^{\bar{n}-j}(X_{\bar{n}}, X_j) \cong \operatorname{Ext}^1(X_{\bar{n}}, X_{\bar{n}-1}) \neq 0$. Thus $\{X_n\}_{n \in \mathbb{N}}$ is a family of non-isomorphic complements of T. In order to show that each $T_j \to X_j \to 0$ is a minimal right $\operatorname{add} T$ approximation of X_j, it is enough to observe that $X_{j-1} \in T^{\perp}$ and $\operatorname{add} X_{j-1} \cap \operatorname{add} T = 0$. $\square$

Notice that Proposition 5.3 can be compared with results in [7,11,12]. In these quoted papers, giving a partial tilting modules M, the long exact sequence whose cokernels are complements of M can be constructed when $\delta(M) = \delta(\Lambda) - 1$. Conversely, Proposition 5.3 can be applied for any value of $\delta(T)$.

Moreover we point out that, in the Wakamatsu case, the long exact sequence giving the complements could be not unique; starting from non-isomorphic complements M_1 and M_2 we could theoretically construct two different sequences. Let us describe better the structure of these sequences:

(1) Each sequence of complements can be continued on the left, until we obtain a complement not generated by T. Indeed, if a sequence starts with a complement M generated by T, then the minimal right $\operatorname{add} T$-approximation of M is an epimorphism. With the same techniques as used in Proposition 5.3, it is easily verified that the kernel of the minimal right approximation is a complement of T. If $\operatorname{pd} M = n$, we cannot repeat this argument more than n times: if

$$\cdots \to T_{-n} \to \cdots \to T_{-2} \xrightarrow{f_{-2}} T_{-1} \xrightarrow{f_{-1}} M \to 0$$

is a long exact sequence where the $\ker f_{-i}$ are complements of T for any i, we would obtain $\operatorname{Ext}^1(M, \ker f_{-1}) \cong \operatorname{Ext}^{n+1}(M, \ker f_{-n-1}) = 0$ and so $M \in \operatorname{add} T$.

(2) Two different sequences don't intersect each other. This follows from the fact that the maps on the sequence are minimal approximations, so they are essentially unique. Since any sequence starts with a complement not generated by T, if two sequences intersect each other, they would start with the same complement. Then they would coincide.

The same argument used in Proposition 5.3 proves that, if a Wakamatsu tilting module of finite projective dimension admits a complement of finite injective dimension, then it is a tilting module.

5.4. Corollary. *Let T be a Wakamatsu tilting module of finite projective dimension. Let M be a complement of T such that* $\operatorname{id} M = n < \infty$. *Then T is a tilting module.*

Proof. We construct a sequence of complements of T starting from $M = N_0$, as in the proof of Proposition 5.3. Then

$$\operatorname{Ext}^1(N_{n+1}, N_n) \cong \operatorname{Ext}^{n+1}(N_{n+1}, N_0) = 0,$$

and so $N_n \in \operatorname{add} T$. Thus T is a tilting module. □

Notice that Proposition 5.3 partially generalizes the result of Buan and Solberg mentioned in Theorem 4.5, that a Wakamatsu tilting module which is an almost complete tilting module has an infinite number of complements, since we have replaced "almost complete" by "partial". To understand what can be said about the other direction, let T be a partial tilting module. Since we do not know in general how the different complements of T are related, we cannot argue as in [6, Theorem 3.6]. Nevertheless, if T admits a complement X_0 with an $\operatorname{add} T$-coresolution, we still obtain that T is actually a Wakamatsu tilting module.

5.5. Proposition. *Let $T \in \operatorname{mod} \Lambda$ be a partial tilting module. If T admits a complement with an $\operatorname{add} T$-coresolution, then T admits an infinite number of non-isomorphic complements. Moreover T is a Wakamatsu tilting module.*

Proof. By assumption, T admits a complement X_0 such that there exists a long exact sequence

$$0 \to X_0 \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} T_2 \xrightarrow{f_3} \cdots$$

where $T_i \in \operatorname{add} T$, f_i are left $\operatorname{add} T$-approximations and $\operatorname{coker} f_i = X_{i+1}$ for $i \geqslant 0$. By an argument similar to that applied in Lemma 5.2, it follows that the modules $\{X_i\}_{i \geqslant 0}$ are non-isomorphic complements of T. Since $T \oplus X_i$ is a tilting module for any $i \geqslant 0$, it follows from Proposition 4.1 that ${}^{\perp}(T \oplus X_i) \cap \mathcal{P}^{<\infty} \subseteq \operatorname{cogen}(T \oplus X_i)$; moreover, it is easy to verify that ${}^{\perp}(T \oplus X_i) \cap \mathcal{P}^{<\infty} \subseteq {}^{\perp}(T \oplus X_{i+1}) \cap \mathcal{P}^{<\infty}$. Let h_0 be a minimal left $\operatorname{add}(T \oplus X_0)$-approximation of Λ; then $K_0 = \operatorname{coker} h_0$ belongs to ${}^{\perp}(T \oplus X_0) \cap \mathcal{P}^{<\infty} \subseteq {}^{\perp}(T \oplus X_1) \cap \mathcal{P}^{<\infty}$. Let us consider the following pushout commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda \arrow[r, "h_0"] \arrow[d, equal] & T_0' \oplus X_0' \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \arrow[r, "g_0"] & T_0' \oplus T_0'' \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
 & & X_1' \arrow[r, equal] \arrow[d] & X_1' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $T_0', T_0'' \in \operatorname{add} T$, $X_0' \in \operatorname{add} X_0$, and $X_1' \in \operatorname{add} X_1$. Since $Q_1 \in {}^{\perp}(T \oplus X_1) \cap \mathcal{P}^{<\infty}$, Q_1 is cogenerated by $T \oplus X_1$. Let h_1 be a left $\operatorname{add}(T \oplus X_1)$-approximation of Q_1. From a diagram analogous to the previous one, we can construct an exact sequence $0 \to Q_1 \xrightarrow{g_1} T_1' \oplus T_1'' \to Q_2 \to 0$ where Q_2 belongs to ${}^{\perp}(T \oplus X_2) \cap \mathcal{P}^{<\infty}$ and so it is cogenerated by $T \oplus X_2$. Repeating this construction, we obtain a long exact sequence

$$0 \to \Lambda \xrightarrow{g_0} \overline{T}_0 \xrightarrow{g_1} \overline{T}_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} \overline{T}_n \xrightarrow{g_{n+1}} \cdots$$

where $\overline{T}_i \in \operatorname{add} T$ and $\operatorname{coker} g_i \in {}^{\perp}T$ for any $i \geqslant 0$. It follows that T is a Wakamatsu tilting module. $\square$

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